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The modulation of weakly non-linear dispersive waves near the marginal state of instability

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Abstract. The modulation of a one-dimensional weakly non-linear purely dispersive quasi-monochromatic wave (the carrier) is usually governed by the non-linear Schrödinger (NS) equation. The critical wavenumber for which the carrier is marginally modulationally unstable is determined by the condition that the product of the coefficients of the non-linear and dispersive terms in the NS equation is zero. However, near this marginal state the assumptions that lead to the NS equation are invalid and a modified form of the NS equation that involves higher-order non-linearities is appropriate. This modified NS equation is here derived formally for a general system involving a single dependent variable and a revised form of the instability criterion is obtained. The results are illustrated by considering a particular system described by a generalised Korteweg-de Vries equation.

1. Introduction

The amplitude modulation of a one-dimensional weakly non-linear quasi-monochromatic purely dispersive wave (the carrier wave) may usually be described by the non-linear Schrödinger (NS) equation

$$\mathrm{i}\frac{\partial\phi}{\partial\tau_2}+p\frac{\partial^2\phi}{\partial\xi_1^2}=q|\phi|^2\phi \tag{1.1}$$

where $\tau_2=\varepsilon^2 t$, $\xi_1=\varepsilon(x-V_g t)$, x and t are space and time coordinates respectively, ε is a small parameter, ϕ, V_g and k_0 are the complex amplitude, group velocity and wavenumber of the carrier wave, respectively, and $p=\frac{1}{2}\,\mathrm{d}V_g/\mathrm{d}k_0$ and q are real functions of k_0. The NS equation has been derived for many physical systems using various perturbation techniques such as the Krylov-Bogoliubov-Mitropolsky (KBM) method, the reductive perturbation method, and the derivative expansion method. Jeffrey and Kawahara (1982) give a representative selection of references.

The NS equation has a plane wave solution of constant amplitude that is unstable if

$$pq<0. \tag{1.2}$$

It follows that (1.2) is the condition for the modulational instability of the carrier wave. The instability is due to a non-linear resonance mechanism in which side-band perturbations to the carrier wave reinforce the harmonics of the carrier (Benjamin and Feir 1967).

If pq has no real zero then the carrier is either modulationally stable or unstable for all wavenumbers according to whether $pq>0$ or $pq<0$ respectively. For many physical systems pq has just one real zero, at some critical wavenumber k_c, and furthermore $pq \sim k_0 - k_c$ for k_0 near k_c. The condition (1.2) may then be written as $k_0 < k_c$ or $k_0 > k_c$ depending on whether pq is increasing or decreasing at $k_0 = k_c$. It appears then that the carrier wave is marginally modulationally unstable at $k_0 = k_c$.

In the present notation the side-bands have a width of $O(\varepsilon)$ and lead to the dispersive term involving p in (1.1). The NS equation is derived by balancing the effects of weak non-linearity and small side-band width. In order to do this it is usually assumed that the non-linearity is also of $O(\varepsilon)$; this then leads to (1.1) with the coefficients p and q, in the dispersive and non-linear terms respectively, both of $O(1)$.

In this paper we shall assume that $p(k_c) \neq 0$ and moreover that $p(k_c)$ is of $O(1)$, so that when k_0 is near k_c, q is small. Thus an inconsistency must arise in any attempt to achieve the aforementioned balance and consequently the NS equation is not appropriate near the marginal state. Kakutani and Michihiro (1983) argued that, near the marginal state, a different ordering should be used to intensify the effects of the non-linearity; this leads to a new governing equation for ϕ to replace the NS equation, and to a revised modulational instability criterion to replace (1.2). To illustrate this argument Kakutani and Michihiro considered the modulation of Stokes waves (i.e. gravity waves on water of uniform depth) near the marginal state. They intensified the effect of the non-linearity by assuming that the non-linearity is of $O(\varepsilon^{1/2})$ instead of $O(\varepsilon)$; by this means they derived a governing equation for ϕ of the form

$$\mathrm{i}\frac{\partial \phi}{\partial \tau_2} + p\frac{\partial^2 \phi}{\partial \xi_1^2} = q_1|\phi|^2\phi + q_2|\phi|^4\phi + \mathrm{i}q_3\phi\frac{\partial}{\partial \xi_1}|\phi|^2 + \mathrm{i}q_4|\phi|^2\frac{\partial \phi}{\partial \xi_1} \tag{1.3}$$

where q_1, q_2, q_3, q_4 are real functions of the wavenumber k_0, which is assumed to be such that $k_0 - k_c$ is of $O(\varepsilon)$. We shall refer to (1.3) as the modified non-linear Schrödinger (MNS) equation. Kakutani and Michihiro also derived a revised modulational instability criterion of the form

$$pq_1 - r < 0 \tag{1.4}$$

where r depends on k_0 and the amplitude and bandwidth of the carrier wave. In effect (1.4) shows that the correct critical wavenumber for marginal modulational instability is slightly different from k_c.

The modulation of Stokes waves has also been considered by Johnson (1977). He obtained an equation slightly different from (1.3) and a corresponding modulational instability criterion.

The NS equation can be derived in a purely formal way without reference to a particular physical system (see Jeffrey and Kawahara 1982, § 3.2.1, for example). In § 2 we show how to derive the MNS equation in a similar way using the derivative expansion procedure (Kawahara 1973). In § 3 we derive (1.4) using a method different from the one used by Kakutani and Michihiro (1983). In §§ 4 and 5 we consider, as an example, a system described by a mixed Korteweg-de Vries (KdV) and modified KdV equation. In § 4 the NS equation and the modulational instability criterion are derived on the assumption that k_0 is not near k_c. In § 5 we consider the marginal state and derive the MNS equation and the revised modulational instability criterion.

2. A formal derivation of the modified non-linear Schrödinger equation

In this section we show that the MNS equation can be derived formally for any non-linear dispersive system in which a single dependent variable u satisfies an equation of the form

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)u = N(u) \tag{2.1}$$

where L is a linear operator involving the differential operators $\partial/\partial t$ and $\partial/\partial x$ and $N(u)$ represents all the non-linear terms.

First consider the linearised problem

$$Lu = 0. \tag{2.2}$$

Suppose we require a uniform monochromatic wavetrain solution to (2.2) of the form

$$u = A_0 \exp[i(kx - \omega t)] \tag{2.3}$$

where ω is the angular frequency and A_0 is a non-zero complex constant. Insertion of (2.3) into (2.2) gives

$$L(-i\omega, ik)u = 0$$

from which it follows that

$$D(\omega, k)A_0 = 0 \tag{2.4}$$

where D is a real function of ω and k. As $A_0 \neq 0$

$$D(\omega, k) = 0$$

and this is the linear dispersion relation which, for a purely dispersive system, is satisfied by real values of ω and k. A general solution of (2.2) is just a superposition of solutions like (2.3). In particular the solution to the linearised problem that represents a slowly modulated wavetrain with most of the energy in wavenumbers near to some constant value k_0 is the quasi-monochromatic plane wave

$$u = A \exp[i(k_0 x - \omega_0 t)] \tag{2.5}$$

where $\omega_0 = \omega(k_0)$ is determined from the linear dispersion relation and A is a slowly varying function of x, t (see Jeffrey and Kawahara 1982, § 3.2.1, for example).

Our aim is to derive a governing equation for A. We obtain this by first seeking an appropriate generalisation of (2.4). In order to distinguish between the fast oscillations of the wavetrain (2.5) and the slow modulations we may use the derivative expansion procedure (Kawahara 1973). We introduce the extended set of independent variables

$$t_i = \varepsilon^i t \qquad x_i = \varepsilon^i x \qquad (i = 0, 1, 2, \ldots, N)$$

where ε is a small parameter characterising the slow modulation. Only those variables with $i = 0, 1, 2$ occur in the equations in this paper, so it is sufficient to take $N = 2$. Had we wished to work to higher order we would have needed further variables. Thus defined t_0, x_0 are the variables appropriate to 'fast' variations and t_1, x_1, t_2, x_2 are the 'slow' variables. The differential operators can now be expressed as the derivative expansions

$$\frac{\partial}{\partial t} \equiv -\omega_0 \frac{\partial}{\partial \theta} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} \qquad \frac{\partial}{\partial x} \equiv k_0 \frac{\partial}{\partial \theta} + \varepsilon \frac{\partial}{\partial x_1} + \varepsilon^2 \frac{\partial}{\partial x_2}$$

where $\theta = k_0 x_0 - \omega_0 t_0$ is, to lowest order, the phase of the fast oscillations of the wavetrain.

We now assume that the appropriate generalisation of (2.4) that takes into account both modulation and non-linear effects is

$$D\left(\omega_0+\mathrm{i}\varepsilon\frac{\partial}{\partial t_1}+\mathrm{i}\varepsilon^2\frac{\partial}{\partial t_2},\ k_0-\mathrm{i}\varepsilon\frac{\partial}{\partial x_1}-\mathrm{i}\varepsilon^2\frac{\partial}{\partial x_2},\ |A|^2\right)A=0 \tag{2.6}$$

corresponding to a solution $u=A\,\mathrm{e}^{\mathrm{i}\theta}$ and a non-linear dispersion relation

$$D(\omega, k, |A|^2)=0. \tag{2.7}$$

First suppose that the non-linearity is of $O(\varepsilon)$, so that we may write $A=\varepsilon\phi$. We expand (2.6) in a Taylor series about $(\omega_0, k_0, 0)$. At the lowest order, namely $O(\varepsilon)$, we obtain the linear dispersion relation

$$D(\omega_0, k_0, 0)=0. \tag{2.8}$$

Noting that the dispersion relations (2.8) and (2.7) may be rearranged to give $\omega_0=\omega_0(k_0)$ and $\omega=\omega(k, |A|^2)$, respectively, we may derive the relationships

$$V_g=\frac{\mathrm{d}\omega_0}{\mathrm{d}k_0}=-\left(\frac{\partial D}{\partial k}\right)_0\left(\frac{\partial D}{\partial \omega}\right)_0^{-1}=\left(\frac{\partial \omega}{\partial k}\right)_0 \tag{2.9}$$

$$\frac{\mathrm{d}V_g}{\mathrm{d}k_0}=\frac{\mathrm{d}^2\omega_0}{\mathrm{d}k_0^2}=-\left[\left(V_g^2\frac{\partial^2 D}{\partial\omega^2}+2V_g\frac{\partial^2 D}{\partial\omega\,\partial k}+\frac{\partial^2 D}{\partial k^2}\right)_0\left(\frac{\partial D}{\partial\omega}\right)_0^{-1}\right]=\left(\frac{\partial^2\omega}{\partial k^2}\right)_0 \tag{2.10}$$

where $(\ )_0$ denotes evaluation at $\omega=\omega_0$, $k=k_0$, $|A|^2=0$.

It is convenient at this stage to introduce new slow variables

$$\tau_i=t_i \qquad \xi_i=x_i-V_g t_i \qquad (i=1,2)$$

corresponding to a reference frame moving with velocity V_g.

Substituting these new variables into (2.6) we obtain at $O(\varepsilon^2)$

$$\partial\phi/\partial\tau_1=0 \tag{2.11}$$

where we have used (2.9). Making use of (2.10) and (2.11), we obtain at $O(\varepsilon^3)$ the NS equation (1.1) with q given by

$$q=-\left(\frac{\partial D}{\partial|A|^2}\right)_0\left(\frac{\partial D}{\partial\omega}\right)_0^{-1}. \tag{2.12}$$

An explicit expression for q can be obtained only with knowledge of the non-linear dispersion relation (2.7), and this in turn depends on the structure of the non-linear term $N(u)$ in (2.1).

In deriving (1.1) at $O(\varepsilon^3)$ from the expansion of (2.6) we have implicitly assumed that q is of $O(1)$. However, as pointed out in § 1, if k_0 is near k_c then q is in fact very small and (1.1) is not the appropriate governing equation. In this case to balance the non-linear and side-band effects in the expansion of (2.6) we intensify the effect of the non-linearity by writing $A=\varepsilon^{1/2}\phi$, and assume that $q=\varepsilon q_1$ for k_0 near k_c, where q_1 is of $O(1)$ and q is given by (2.12). At $O(\varepsilon^{1/2})$ and $O(\varepsilon^{3/2})$ we again obtain (2.8) and (2.11), respectively. At $O(\varepsilon^{5/2})$ we obtain

$$\begin{aligned}\left(\frac{\partial D}{\partial\omega}\right)_0&\left(\mathrm{i}\frac{\partial\phi}{\partial\tau_2}+\frac{1}{2}\frac{\mathrm{d}V_g}{\mathrm{d}k_0}\frac{\partial^2\phi}{\partial\xi_1^2}\right)\\ &=\left(\frac{\partial D}{\partial\omega}\right)_0 q_1|\phi|^2\phi-\frac{1}{2}\left(\frac{\partial^2 D}{\partial(|A|^2)^2}\right)_0|\phi|^4\phi\\ &\quad+\mathrm{i}V_g\left(\frac{\partial^2 D}{\partial\omega\,\partial|A|^2}\right)_0\frac{\partial}{\partial\xi_1}|\phi|^2\phi+\mathrm{i}\left(\frac{\partial^2 D}{\partial k\,\partial|A|^2}\right)_0\frac{\partial}{\partial\xi_1}|\phi|^2\phi.\end{aligned} \tag{2.13}$$

In the last two terms the operations $\partial/\partial\xi_1$ and $|\phi|^2$ acting on ϕ do not commute, nor do they necessarily occur in the order shown. The correct order can be determined only if the structure of the non-linear term $N(u)$ in (2.1) is known. With this proviso, (2.13) represents the desired MNS equation, which may be written in the form (1.3) with

$$q_2 = -\frac{1}{2}\left(\frac{\partial^2 D}{\partial(|A|^2)^2}\right)_0\left(\frac{\partial D}{\partial\omega}\right)_0^{-1}$$

$$q_4 = V_g\left(\frac{\partial^2 D}{\partial\omega\,\partial|A|^2}\right)_0\left(\frac{\partial D}{\partial\omega}\right)_0^{-1} + \left(\frac{\partial^2 D}{\partial k\,\partial|A|^2}\right)_0\left(\frac{\partial D}{\partial\omega}\right)_0^{-1}$$

and q_3 equal to q_4 or to one or other of the terms in q_4. Note that we may use (2.7) to show that q_1, q_2 and q_4 may also be written

$$\varepsilon q_1 = \left(\frac{\partial\omega}{\partial|A|^2}\right)_0 \tag{2.14}$$

$$q_2 = \frac{1}{2}\left(\frac{\partial^2\omega}{\partial(|A|^2)^2}\right)_0 + \mathrm{O}(\varepsilon) \tag{2.15}$$

$$q_4 = -\left(\frac{\partial^2\omega}{\partial k\,\partial|A|^2}\right)_0 + \mathrm{O}(\varepsilon). \tag{2.16}$$

3. The instability criterion

Taniuti and Yajima (1969) derived the modulational instability criterion (1.2) for a carrier whose complex amplitude is governed by the NS equation (1.1); they used a method that involves real perturbations to a real amplitude and a real phase. This method was used by Kakutani and Michihiro (1983) to establish the modulational instability criterion (1.4) near the marginal state for a carrier whose complex amplitude is governed by the MNS equation (1.3). Using an alternative method based on that of Stuart and DiPrima (1978) which involves a complex perturbation to a complex amplitude, Dodd *et al* (1982) rederived (1.2). We use a generalisation of their method to rederive (1.4).

Equation (1.3) has a plane wave solution of constant amplitude of the form

$$\phi = \phi_0 \exp[\mathrm{i}(\kappa\xi_1 - \Omega\tau_2)] \tag{3.1}$$

where ϕ_0 is a complex constant and κ, Ω are real constants satisfying

$$\Omega = p\kappa^2 + q_1|\phi_0|^2 + q_2|\phi_0|^4 - \kappa q_4|\phi_0|^2.$$

The quantity κ may be interpreted as a measure of the spread of wavenumbers about the dominant wavenumber k_0 in the carrier wave (2.5), as may be seen as follows. Inserting (3.1) into (2.5) we obtain (2.3) with $A_0 = \varepsilon^{1/2}\phi_0$, $k = k_0 + \varepsilon\kappa$ and

$$\omega = \omega_0 + \varepsilon\kappa V_g + \varepsilon^2\Omega \tag{3.2}$$

where we have used $\xi_1 = \varepsilon(x - V_g t)$ and $\tau_2 = \varepsilon^2 t$. Inserting (2.9), (2.10) and (2.14)-(2.16) into (3.2), and noting that $\varepsilon|\phi_0|^2 = |A|^2$, we obtain

$$\omega = \omega_0 + \left(\frac{\partial\omega}{\partial k}\right)_0 (k - k_0) + \frac{1}{2}\left(\frac{\partial^2\omega}{\partial k^2}\right)_0 (k-k_0)^2 + \left(\frac{\partial\omega}{\partial|A|^2}\right)_0 |A|^2 + \frac{1}{2}\left(\frac{\partial^2\omega}{\partial(|A|^2)^2}\right)_0 |A|^4$$

$$+\left(\frac{\partial^2\omega}{\partial k\,\partial|A|^2}\right)_0(k-k_0)|A|^2+\mathrm{O}(\varepsilon^3)$$

which is just the Taylor series expansion about $(k_0, 0)$ of $\omega=\omega(k,|A|^2)$.

To investigate the modulational instability of the carrier we consider a perturbation of (3.1) of the form

$$\phi=[1+B(\tau_2,\xi_1)]\phi_0\exp[\mathrm{i}(\kappa\xi_1-\Omega\tau_2)] \tag{3.3}$$

where B is complex. Inserting (3.3) into (1.3) and linearising with respect to B we obtain

$$\mathrm{i}\frac{\partial B}{\partial\tau_2}+p\left(\frac{\partial^2 B}{\partial\xi_1^2}+2\mathrm{i}\kappa\frac{\partial B}{\partial\xi_1}\right)$$
$$=(q_1+2q_2|\phi_0|^2-\kappa q_4)|\phi_0|^2(B+B^*)+\mathrm{i}q_3|\phi_0|^2\frac{\partial}{\partial\xi_1}(B+B^*)+\mathrm{i}q_4|\phi_0|^2\frac{\partial B}{\partial\xi_1} \tag{3.4}$$

where * denotes the complex conjugate. We seek a solution to (3.4) in the form

$$B=B_1\exp[\mathrm{i}(\tilde{k}\xi_1-\tilde{\omega}\tau_2)]+B_2\exp[-\mathrm{i}(\tilde{k}\xi_1-\tilde{\omega}^*\tau_2)] \tag{3.5}$$

where B_1, B_2 are complex constants, $\tilde{k}$ is a real wavenumber and $\tilde{\omega}$ may be complex. Substitution of (3.5) into (3.4) leads to the following linear homogeneous system for B_1 and B_2^*:

$$(a-b-c)B_1+(d-c)B_2^*=0$$
$$(d+c)B_1+(a+b+c)B_2^*=0$$

where

$$a=\tilde{\omega}-2\kappa p\tilde{k}+(q_3+q_4)\tilde{k}|\phi_0|^2 \qquad b=p\tilde{k}^2$$
$$c=(q_1+2q_2|\phi_0|^2-\kappa q_4)|\phi_0|^2 \qquad d=q_3\tilde{k}|\phi_0|^2.$$

(The second of these equations is in fact the complex conjugate of the relation that arises directly.) The condition for a non-trivial solution gives the dispersion relation

$$a^2=b^2+2bc+d^2.$$

If the right-hand side is negative then the angular frequency $\tilde{\omega}$ will be complex and the perturbations will grow. In this case

$$pq_1-r<-p^2\tilde{k}^2/2|\phi_0|^2<0$$

where

$$r=p\kappa q_4-(q_3^2+4pq_2)|\phi_0|^2/2 \tag{3.6}$$

and wavenumbers $\tilde{k}$ such that

$$0<\tilde{k}<|\phi_0|[2(r-pq_1)]^{1/2}/|p|$$

will be unstable. Thus the modulational instability criterion is (1.4) in agreement with equation (2.8) of Kakutani and Michihiro (1983).

Unlike the condition (1.2) from the NS equation, (1.4) depends via r upon $|\phi_0|$, the amplitude of the carrier and κ, the wavenumber spread. In turn these quantities will affect the behaviour of the system near the critical state given by $pq_1=0$ for which $k_0=k_c$. Let us define the subcritical state as that for which $pq_1>0$ and the supercritical state as that for which $pq_1 < 0$. From (1.4) we deduce that if $r > 0$ then all supercritical waves are unstable and that subcritical waves with $r > pq_1 > 0$ are unstable. On the other hand, if $r < 0$ then all subcritical waves are stable and supercritical waves with $r < pq_1 < 0$ are stable. Kakutani and Michihiro (1983) show that for a Stokes wave, $\kappa = 0$ and $r > 0$, so that a Stokes wave is in the former category. Johnson (1977), however, finds that $r < 0$ and deduces that a Stokes wave is in the latter category.

4. An example of the non-linear Schrödinger equation

In this section and the next we apply the derivative expansion procedure to an equation that is particularly simple but physically relevant, namely

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta u^2 \frac{\partial u}{\partial x} + \gamma \frac{\partial^3 u}{\partial x^3} = 0 \tag{4.1}$$

where α, β, γ are real constants. We assume that $\gamma \neq 0$. With $\alpha \neq 0$, $\beta = 0$, (4.1) is the KdV equation and with $\alpha = 0$, $\beta \neq 0$, it is the modified KdV equation.

Examples of the occurrence of (4.1) can be found in Kakutani and Yamasaki (1978) and Watanabe (1984). The former authors considered the propagation of long gravity waves on a stably stratified two-layer fluid. Near a critical thickness ratio the governing equation for the elevation of the interface in the slow mode is (4.1). Watanabe (1984) investigated the long wavelength approximation of the system of equations governing the propagation of ion acoustic waves in a plasma comprising electrons and positive and negative ions. Near a critical density of negative ions the electrostatic potential is governed by (4.1)†. In subsequent work on the same physical system Saito *et al* (1984) used the reductive perturbation method to derive from (4.1) the NS equation (1.1) for the complex wave amplitude. Implicit in their derivation was the assumption that k_0 was not near k_c. We shall recover the detailed form of q in this case before going on in § 5 to consider what happens when k_0 is near k_c.

Equation (4.1) can be written in the form (2.1) with

$$L \equiv \frac{\partial}{\partial t} + \gamma \frac{\partial^3}{\partial x^3}$$

and

$$N(u) \equiv -\frac{\partial}{\partial x}\left(\frac{\alpha}{2} u^2 + \frac{\beta}{3} u^3\right).$$

Introducing the extended set of independent variables defined in § 2 we can write

$$L = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + O(\varepsilon^3) \tag{4.2}$$

where

$$L_0 \equiv -\omega_0 \frac{\partial}{\partial \theta} + \gamma k_0^3 \frac{\partial^3}{\partial \theta^3}$$

$$L_1 \equiv \frac{\partial}{\partial \tau_1} - V_g \frac{\partial}{\partial \xi_1}\left(1 + \frac{\partial^2}{\partial \theta^2}\right)$$

† There is an error in the coefficient corresponding to our α in equation (30) of Watanabe (1984) and in equation (6) of Saito *et al* (1984). Their coefficient should be divided by the small parameter labelled ε in their notation. There is no such error in equation (5.6) of Kakutani and Yamasaki (1978).

$$L_2 \equiv \frac{\partial}{\partial \tau_2} - V_g \frac{\partial}{\partial \xi_2}\left(1+\frac{\partial^2}{\partial \theta^2}\right) - p\frac{\partial^2}{\partial \xi_1^2}\frac{\partial}{\partial \theta}.$$

When the non-linearity is assumed to be of $O(\varepsilon)$, u may be written

$$u = \sum_{n=1}^{3} \varepsilon^n u_n(\theta, \tau_1, \xi_1, \tau_2, \xi_2) + O(\varepsilon^4). \tag{4.3}$$

We may now write

$$N(u) = -\varepsilon^2 N_2 - \varepsilon^3 N_3 + O(\varepsilon^4) \tag{4.4}$$

where N_2 and N_3 are given in the appendix. Substituting (4.2)-(4.4) into (2.1) and equating like powers of ε we obtain the hierarchy of equations

$$O(\varepsilon^n):\ L_0 u_n = \begin{cases} 0 & n=1 \\ -\sum\limits_{j=1}^{n-1} L_j u_{n-j} - N_n & n=2,3. \end{cases} \tag{4.5}$$

We may now solve these equations in turn, noting that for each $n>1$ up to two non-secular conditions may be obtained by setting both the θ independent terms and the coefficient of $e^{i\theta}$ to zero on the right-hand side of (4.5), if they are not already identically zero.

We assume the following quasi-monochromatic wave as the solution of the $O(\varepsilon)$ equation in (4.5):

$$u_1 = \phi(\tau_1, \xi_1, \tau_2, \xi_2)\, e^{i\theta} + \text{cc} \tag{4.6}$$

where ϕ is a complex function and ω_0, k_0 satisfy the linear dispersion relation corresponding to (2.8), namely

$$\omega_0 + \gamma k_0^3 = 0. \tag{4.7}$$

Here, and subsequently, 'cc' is used to mean 'the complex conjugate of all the preceding terms'. We note that for $n>1$ the homogeneous solutions to (4.5) may be included in (4.6) by suitably redefining ϕ, and that ω_0 may be eliminated from the u_n by means of (4.7). Also from (4.7) we have $V_g = -3\gamma k_0^2$ and $p = -3\gamma k_0$.

At $O(\varepsilon^2)$ there is one non-secular condition (from the coefficient of $e^{i\theta}$), namely

$$\partial\phi/\partial\tau_1 = 0 \tag{4.8}$$

(cf (2.11)), and then

$$u_2 = \frac{\alpha}{6\gamma k_0^2}\phi^2 e^{2i\theta} + \text{cc} + \psi_2 \tag{4.9}$$

where ψ_2 is real and independent of θ. We assume that ψ_2 depends on τ_1 and ξ_1 through ϕ and ϕ^* only and hence, in view of (4.8), that it is independent of τ_1.

At $O(\varepsilon^3)$ there are two non-secular conditions. From the θ independent terms we obtain

$$\frac{\partial\psi_2}{\partial\tau_1} + 3\gamma k_0^2 \frac{\partial\psi_2}{\partial\xi_1} = -\alpha\frac{\partial}{\partial\xi_1}|\phi|^2 \tag{4.10}$$

which may be integrated to give

$$\psi_2 = -\frac{\alpha}{3\gamma k_0^2}|\phi|^2 + \nu_2 \tag{4.11}$$

where ν_2 is an arbitrary real function of τ_2 and ξ_2. If we assume that $u=0$ in the unperturbed state, i.e. when $\phi=0$ so that there is no wave, then, following Kakutani and Michihiro (1983), we may set $\nu_2=0$. The other non-secular condition now gives the NS equation (1.1) with

$$q=\beta k_0-\frac{\alpha^2}{6\gamma k_0}. \tag{4.12}$$

The modulational instability criterion (1.2) is

$$pq\equiv -3k_0^2\left(\beta\gamma-\frac{\alpha^2}{6k_0^2}\right)<0. \tag{4.13}$$

Clearly if $\beta\gamma\leqslant 0$ (as in Kakutani and Yamasaki 1978) there is stability for all k_0. However, if $\beta\gamma>0$ (as in Watanabe 1984), (4.13) may be written $k_0>k_c$, where $k_c=(\alpha^2/6\beta\gamma)^{1/2}$. In the next section we shall assume that $\beta\gamma>0$ and investigate the behaviour of the system when k_0 is near k_c.

5. Derivation of the modified non-linear Schrödinger equation

Following Kakutani and Michihiro (1983) we intensify the non-linear effects by writing

$$u=\sum_{n=1}^{6}\varepsilon^{n/2}u_n(\theta,\tau_1,\xi_1,\tau_2,\xi_2)+O(\varepsilon^{7/2}) \tag{5.1}$$

so that

$$N(u)=-\sum_{n=1}^{6}\varepsilon^{n/2}N_n+O(\varepsilon^{7/2}) \tag{5.2}$$

where the N_n are given in the appendix. Substituting (4.2), (5.1) and (5.2) into (2.1) and equating like powers of ε we obtain the hierarchy of equations

$$O(\varepsilon^{n/2})\colon\ L_0u_n=\begin{cases}-N_n & n=1,2\\ -\sum_{j=1}^{m}L_ju_{n-2j}-N_n & n=3,4,5,6\end{cases} \tag{5.3}$$

where

$$m=\begin{cases}(n-2)/2 & n \text{ even}\\ (n-1)/2 & n \text{ odd.}\end{cases}$$

We may solve these equations in turn and obtain non-secular conditions in the same way as for the hierarchy (4.5).

At $O(\varepsilon^{1/2})$ u_1 is given by (4.6). At $O(\varepsilon)$ there are no non-secular conditions and u_2 is given by (4.9). Again we assume that ψ_2 is independent of τ_1. At $O(\varepsilon^{3/2})$ there is one non-secular condition (from the coefficient of $e^{i\theta}$), namely

$$i\frac{\partial\phi}{\partial\tau_1}=k_0\left[\left(\frac{\alpha^2}{6\gamma k_0^2}+\beta\right)|\phi|^2\phi+\alpha\psi_2\phi\right] \tag{5.4}$$

and then

$$u_3=\frac{1}{24\gamma k_0^2}\left(\frac{\alpha^2}{2\gamma k_0^2}+\beta\right)\phi^3\,e^{3i\theta}+\text{cc}+\psi_3$$

where ψ_3 is real and independent of θ.

At $O(\varepsilon^2)$ there are two non-secular conditions. From the coefficient of $e^{i\theta}$ we find that $\psi_3=0$. The other condition is just (4.10). This may be integrated to give the expression (4.11) for ψ_2, where we have used the fact that $|\phi|^2$ is independent of τ_1, as can be shown by combining (5.4) with its complex conjugate. As before we set $\nu_2=0$. Insertion of (4.11) into (5.4) gives

$$\mathrm{i}\frac{\partial\phi}{\partial\tau_1}=q(k_0)|\phi|^2\phi \tag{5.5}$$

where q is given by (4.12). If we assume that $\Delta k=k_0-k_c$ is of $O(\varepsilon)$ then $q=\varepsilon q_1$, where q_1 is of $O(1)$ and is given approximately by

$$q_1=\frac{\Delta k}{\varepsilon}\left(\frac{\mathrm{d}q}{\mathrm{d}k_0}\right)_{k_0=k_c}=\frac{2\beta\Delta k}{\varepsilon}. \tag{5.6}$$

Hence at $O(\varepsilon^{3/2})$, (5.5) becomes

$$\partial\phi/\partial\tau_1=0 \tag{5.7}$$

(cf (4.8)) and the right-hand side of (5.5) is shifted to the corresponding non-secular condition at $O(\varepsilon^{5/2})$. Now the solution at $O(\varepsilon^2)$ is

$$u_4=\frac{\alpha}{72\gamma^2k_0^4}\left[\left(\frac{\alpha^2}{6\gamma k_0^2}+\beta\right)\phi^4\,e^{4i\theta}+\left(\beta-\frac{5\alpha^2}{6\gamma k_0^2}\right)|\phi|^2\phi^2\,e^{2i\theta}\right]+\frac{i\alpha\phi}{3\gamma k_0^3}\frac{\partial\phi}{\partial\xi_1}e^{2i\theta}+\mathrm{cc}+\psi_4$$

where ψ_4 is real and independent of θ. We assume that ψ_4 is independent of τ_1.

At $O(\varepsilon^{5/2})$ there is one non-secular condition (from the coefficient of $e^{i\theta}$), namely

$$\mathrm{i}\frac{\partial\phi}{\partial\tau_2}+p\frac{\partial^2\phi}{\partial\xi_1^2}=q_1|\phi|^2\phi+\alpha k_0\psi_4\phi+m_2|\phi|^4\phi+\mathrm{i}m_3\phi\frac{\partial}{\partial\xi_1}|\phi|^2+\mathrm{i}m_4|\phi|^2\frac{\partial\phi}{\partial\xi_1} \tag{5.8}$$

where

$$m_2=\frac{1}{24\gamma k_0}\left(\beta^2+\frac{7\alpha^2\beta}{3\gamma k_0^2}-\frac{7\alpha^4}{36\gamma^2k_0^4}\right)\qquad m_3=\frac{\alpha^2}{6\gamma k_0^2}-\beta\qquad m_4=\frac{\alpha^2}{2\gamma k_0^2}-\beta.$$

It is not necessary to solve for u_5, but merely to observe that it involves θ independent terms and terms in $e^{\pm3i\theta}$ and $e^{\pm5i\theta}$.

At $O(\varepsilon^3)$ the non-secular condition from the θ independent terms gives

$$\frac{\partial\psi_4}{\partial\tau_1}+3\gamma k_0^2\frac{\partial\psi_4}{\partial\xi_1}=\frac{\alpha}{3\gamma k_0^2}\left[\frac{\partial}{\partial\tau_2}|\phi|^2+\left(\beta-\frac{\alpha^2}{4\gamma k_0^2}\right)\frac{\partial}{\partial\xi_1}|\phi|^4\right]. \tag{5.9}$$

From (5.8) and its complex conjugate it is easily shown that

$$\mathrm{i}\frac{\partial}{\partial\tau_2}|\phi|^2+p\frac{\partial}{\partial\xi_1}\left(\phi^*\frac{\partial\phi}{\partial\xi_1}-\phi\frac{\partial\phi^*}{\partial\xi_1}\right)=\frac{\mathrm{i}}{2}(2m_3+m_4)\frac{\partial}{\partial\xi_1}|\phi|^4. \tag{5.10}$$

Elimination of the τ_2 derivative between (5.9) and (5.10) and an integration with respect to ξ_1 gives

$$\alpha k_0\psi_4\phi=\alpha k_0\nu_4\phi+n_2|\phi|^4\phi+\mathrm{i}n_3\phi\frac{\partial}{\partial\xi_1}|\phi|^2+\mathrm{i}n_4|\phi|^2\frac{\partial\phi}{\partial\xi_1} \tag{5.11}$$

where ν_4 is an arbitrary real function of τ_2 and ξ_2 that we set to zero just as we set $\nu_2 = 0$ in (4.11) and

$$n_2 = \frac{\alpha^2}{18\gamma^2 k_0^3}\left(\frac{\alpha^2}{3\gamma k_0^2} - \beta\right) \qquad n_3 = \frac{\alpha^2}{3\gamma k_0^2} \qquad n_4 = -2n_3.$$

Substitution of (5.11) into (5.8) now gives the desired MNS equation (1.3) with q_1 given by (5.6) and $q_i = m_i + n_i$ $(i = 2, 3, 4)$.

As we are assuming that k_0 is near to k_c, p, q_2, q_3 and q_4 may be approximated by their values at $k_0 = k_c$, namely

$$p = -3\gamma k_c \qquad q_2 = 2\beta^2/3\gamma k_c \qquad q_3 = 2\beta \qquad q_4 = -2\beta.$$

Using these values, we find from (3.6) that $r = 6\beta\gamma k_c R$, where $R = \kappa + \beta|\phi_0|^2/3\gamma k_c$. The modulational instability criterion (1.4) may now be written

$$k_0 > k_c - \varepsilon R$$

which, as expected, is a small correction to (4.13) written in the form $k_0 > k_c$. The conclusions reached in § 3 may now be stated for the particular system governed by (4.1). As we are assuming that $\beta\gamma > 0$, $r > 0$ implies $R > 0$, and then those subcritical waves with $k_c - \varepsilon R < k_0 < k_c$ are unstable. On the other hand, if $r < 0$, so that $R < 0$, then those supercritical waves with $k_c < k_0 < k_c - \varepsilon R$ are stable.

6. Concluding remarks

In § 2 we showed that the MNS equation governs the modulations near marginal modulational instability of wavelike solutions to the system (2.1), and in §§ 4 and 5 a physically relevant example of such a system was considered. However, many purely dispersive physical systems are described by the more general class of quasi-linear partial differential equations

$$A(U)\frac{\partial U}{\partial t} + B(U)\frac{\partial U}{\partial x} + C(U) = 0 \tag{6.1}$$

where U and C are n-component column vectors and A, B are $n \times n$ matrices. Inoue and Matsumoto (1974) have shown that, under certain restrictions, the modulations of wavelike solutions to (6.1) away from marginal modulational instability are governed by the NS equation. We can show that, under certain restrictions, the modulations near marginal modulational instability are governed by the MNS equation. We hope to report this work in due course.

Kakutani and Michihiro (1983) considered a gravity water wave system which is not of the form (2.1) or (6.1). In that case the calculation of the coefficients q_1, q_2, q_3 and q_4 that are needed for the instability criterion (1.4) was formidable. It is anticipated that this will also be so for any particular example of (6.1). We have found that this is the case for ion acoustic waves in a plasma and we hope to report these results shortly.

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Appendix

The N_n in (4.4) are given by

$$N_2 = k_0 \frac{\partial}{\partial\theta}\left(\frac{\alpha}{2}u_1^2\right)$$

$$N_3 = k_0 \frac{\partial}{\partial\theta}\left(\alpha u_1 u_2 + \frac{\beta}{3}u_1^3\right) + \frac{\partial}{\partial\xi_1}\left(\frac{\alpha}{2}u_1^2\right).$$

The N_n in (5.2) are given by

$$N_1 = 0$$

$$N_2 = k_0 \frac{\partial}{\partial\theta}\left(\frac{\alpha}{2}u_1^2\right)$$

$$N_3 = k_0 \frac{\partial}{\partial\theta}\left(\alpha u_1 u_2 + \frac{\beta}{3}u_1^3\right)$$

$$N_4 = k_0 \frac{\partial}{\partial\theta}\left(\frac{\alpha}{2}(u_2^2 + 2u_1u_3) + \beta u_1^2 u_2\right) + \frac{\partial}{\partial\xi_1}\left(\frac{\alpha}{2}u_1^2\right)$$

$$N_5 = k_0 \frac{\partial}{\partial\theta}\left(\alpha(u_1u_4 + u_2u_3) + \beta(u_1^2u_3 + u_1u_2^2)\right) + \frac{\partial}{\partial\xi_1}\left(\alpha u_1 u_2 + \frac{\beta}{3}u_1^3\right)$$

$$N_6 = k_0 \frac{\partial}{\partial\theta}\left(\frac{\alpha}{2}(u_3^2 + 2u_1u_5 + 2u_2u_4) + \frac{\beta}{3}(u_2^3 + 3u_1^2u_4 + 6u_1u_2u_3)\right)$$
$$+ \frac{\partial}{\partial\xi_1}\left(\frac{\alpha}{2}(u_2^2 + 2u_1u_3) + \beta u_1^2u_2\right) + \frac{\partial}{\partial\xi_2}\left(\frac{\alpha}{2}u_1^2\right).$$

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